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On the central extensions of Poisson brackets of hydrodynamic type

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Abstract
Poisson brackets of hydrodynamic type were introduced to construct a theory of conservation system of hydrodynamic type. They lead to the algebraic and differentially geometrical theory of local translationally invariant Lie algebras of first order, Frobenius-type algebras and Novikov algebras. One of the main ways to construct new examples of these algebras and their corresponding Poisson brackets is through the central extensions. In Balinskii A A and Novikov S P (1985 *Sov. Math. Dokl.* **32** 228–31), there is a general theory of the central extensions. In this paper, we give a further detailed study of two (non-trivial) types of Novikov algebras which directly decide an important kind of central extension of the Poisson brackets. We find that the Novikov algebras in the case $\tau = 3$ are in the case $\tau = 0$ and these two cases coincide in dimension $\leqslant 3$. Moreover, they also coincide for the transitive cases in dimension 4.

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1. Introduction

Poisson brackets of hydrodynamic type were introduced and studied in [1–8]:

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_{k=1}^{N} u_x^k b_k^{ij}(u(x))\delta(x-y). \qquad (1.1)$$

Any hydrodynamic-type Hamiltonian $H = \int h(u(x))\,dx$ (i.e. its density does not depend on derivatives) generates a hydrodynamic-type system with Poisson brackets of hydrodynamic type [2]. The coordinates $(u^1, \ldots, u^N)$ in field space are Liouville or physical if the Poisson bracket (1.1) has the form

$$\{u^i(x), u^j(y)\} = (\gamma^{ij}(u) + \gamma^{ji}(u))\delta'(x-y) + \sum_{k=1}^{N} \frac{\partial \gamma^{ij}}{\partial u^k} u_x^k \delta(x-y). \tag{1.2}$$

The physical coordinates play an important role in applications. Any coordinates such that the Poisson bracket (1.1) is linear, are physical. Then the simplest local Lie algebra arising from the (linear) Poisson brackets of hydrodynamic type (1.1) was introduced as follows [1]:

$$g^{ij} = \sum_{k=1}^{N} C_k^{ij} u^k + g_0^{ij} \qquad b_k^{ij} = \text{const} \quad g_0^{ij} = \text{const} \tag{1.3}$$

$$[p, q]_k(z) = b_k^{ij}(p_i(z)q_j'(z) - q_i(z)p_j'(z)) \qquad b_k^{ij} + b_k^{ji} = C_k^{ij} = \partial g^{ij}/\partial u^k. \tag{1.4}$$

It is known from the Jacobi identity that the tensor b_k^{ij} by equation (1.4) defines a local translationally invariant Lie algebra of first order if and only if $\left\{b_k^{ji}\right\}$ is the set of structure constants of a new finite-dimensional algebra B with a bilinear product $(x, y) \to xy$ satisfying

$$e_i e_j = \sum_{k=1}^{N} b_k^{ji} e_k \tag{1.5}$$

$$(x, y, z) = (y, x, z) \tag{1.6}$$

$$(xy)z = (xz)y \tag{1.7}$$

for any $x, y, z \in B$. Here $\{e_1, e_2, \ldots, e_N\}$ is a basis of B and $(x, y, z) = (xy)z - x(yz)$. (Note that we use the left-symmetry here, whereas the right-symmetry was used in [1].)

The algebra B satisfying equations (1.6), (1.7) is called a 'Novikov algebra' by Osborn [9–12]. It also has a close connection to some Hamiltonian operators in the formal variational calculus and some nonlinear partial differential equations, such as KdV equations [13–17]. On the other hand, Novikov algebras are a special class of left-symmetric algebras which only satisfy equation (1.6). Left-symmetric algebras are a class of non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones [18–21].

The commutator of a Novikov algebra (or a left-symmetric algebra) A,

$$[x, y] = xy - yx \tag{1.8}$$

defines a (sub-adjacent) Lie algebra $\mathcal{G} = \mathcal{G}(A)$. Let L_x, R_x denote the left and right multiplication operators respectively, i.e. $L_x(y) = xy$, $R_x(y) = yx$, $\forall x, y \in A$. Then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative. A Novikov algebra is called right-nilpotent or transitive, if every R_x is nilpotent. The transitivity corresponds to the completeness of the affine manifolds in geometry [18, 19]. Moreover, a finite-dimensional Novikov algebra A over an algebraically closed field with characteristic 0 contains a (unique) largest transitive ideal $N(A)$ (called the radical of A), and the quotient algebra $A/N(A)$ is a direct sum of fields [6].

The study of Poisson brackets of hydrodynamic type also relates to certain discussion of geometry and some other algebras such as Frobenius algebras [1–3]. In fact, in [1–8], there is a study of linear Poisson brackets of hydrodynamic type which leads to the beautiful algebraic and differentially geometrical theory of local translationally invariant Lie algebras of first order, Frobenius-type algebras and their non-associative analogues (Novikov algebras), and super-analogues of that theory. There is also an algebraic study on Novikov algebras in [22–25]. However, it still lacks enough examples which hinders the further development of the theory. One of the main ways to construct new examples is through the extension theory, in particular, through the central extensions.

In [1], there is a theory on the central extensions of the above algebras and their corresponding Poisson brackets. In particular, the Lie algebra (1.4) possesses central extensions by means of the simplest cocycles of the type of the Gel'fand–Fuks cocycle for an algebra of vector fields. The 2-cocycles of order τ are given by the formula

$$\gamma_\tau(p, q) = \int \gamma_\tau^{ij} p_i^{(\tau)} q_j \, \mathrm{d}x = -\gamma_\tau(q, p). \tag{1.9}$$

These cocycles generate additions to the Poisson brackets (1.1) of the form $\gamma_\tau^{ij}\delta^{(\tau)}(x-y)$. Such cocycles are possible for $\tau \leqslant 3$. Furthermore, there is a theorem in [1] as follows: the forms $\delta_\tau^{ij}(u) = \left(b_k^{ji} + (-1)^{\tau+1} b_k^{ij}\right)u^k$ are cocycles and define a central extension of Lie algebra (1.4) for all $(u^1, \ldots, u^N)$ if and only if the Novikov algebra B given by equations (1.6)–(1.7) possesses the following properties:

$\tau = 0$: in B, the identity $[xy + yx, z]/2 = (yx)z - y(xz)$ holds;
$\tau = 1$: the form $\delta_1^{ij}(u)$ always defines a cocycle on the Lie algebra which is a coboundary;
$\tau = 2$: B is associative;
$\tau = 3$: B is such that $x[y, z] + [y, z]x = 0, \forall x, y, z \in B$.

It is obvious that the case $\tau = 1$ is trivial since it holds for any Novikov algebra. And there is a general theory of the associative Novikov algebra (the case $\tau = 2$) in [1]. Thus, a natural question is what kind of Novikov algebra satisfies the additional condition appearing in the case $\tau = 0$ or $\tau = 3$.

In this paper, we will give a further detailed study of these two (non-trivial) types of Novikov algebras. It is quite interesting to see that the Novikov algebras in the case $\tau = 3$ are in the case $\tau = 0$. The paper is organized as follows. In section 2, we discuss the Novikov algebras in the case $\tau = 0$. In section 3, we discuss the Novikov algebras in the case $\tau = 3$. In section 4, we give some conclusions based on the discussion in the previous sections.

2. Novikov algebras in the case $\tau = 0$

In this section, let A be a Novikov algebra with the additional condition

$$[xy + yx, z]/2 = (yx)z - y(xz) \qquad \forall x, y, z \in A. \tag{2.1}$$

Let $x = y$, then we have

$$z(xx) = x(xz) \qquad \forall x, z \in A. \tag{2.2}$$

In contrast, in equation (2.2), we replace x by $x + y$, then we have

$$z(xy + yx) - (xy + yx)z = x(yz) + y(xz) - (xy)z - (yx)z. \tag{2.3}$$

By the left-symmetry (equation (1.6)), we can get equation (2.1). Thus, we have the following claim.

Claim 1. *Let A be a Novikov algebra. Then A satisfies equation (2.1) if and only if A satisfies equation (2.2), that is, $R_{x^2} = L_x^2, \forall x \in A$.*

We would like to point out that the above claim depends on the left-symmetry of A which also decides the 'linearization' process from equation (2.2) to equation (2.1).

Corollary 1. *Let A be a Novikov algebra satisfying equation (2.1). If A is transitive, then the sub-adjacent Lie algebra of A is nilpotent.*

In fact, if A is transitive, then every R_x is nilpotent. Hence, by equation (2.2), every L_x is nilpotent, too. Therefore, from Sheuneman's theorem [18] (if every L_x is nilpotent for a left-symmetric algebra, then its sub-adjacent Lie algebra is nilpotent), the sub-adjacent Lie algebra of A is nilpotent.

Example 1. Obviously, every commutative Novikov algebra (i.e. every commutative associative algebra) satisfies equation (2.1). There are exactly two Novikov algebras in dimension 1: $\mathbf{C} = \{\mathbf{C}e|ee = e\}$ and (T0) $\{\mathbf{C}e|ee = 0\}$. Both of them satisfy equation (2.1).

Example 2. The associative algebra A which satisfies

$$(xy)z = x(yz) = 0 \qquad \forall x, y, z \in A \tag{2.4}$$

is a Novikov algebra satisfying equation (2.1).

Example 3. Recall that the (form) characteristic matrix of a Novikov algebra A is defined as

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^n c_{11}^k e_k & \cdots & \sum_{k=1}^n c_{1n}^k e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^n c_{n1}^k e_k & \cdots & \sum_{k=1}^n c_{nn}^k e_k \end{pmatrix} \tag{2.5}$$

where $\{e_i\}$ is a basis of A and $e_i e_j = \sum_{k=1}^n c_{ij}^k e_k$. From the classification of Novikov algebras in dimension 2 over the complex field given in [22], the classification of two-dimensional complex commutative associative algebras is given as follows:

$$(T1)\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (T2)\begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix} \quad (N1)\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \quad (N2)\begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix} \quad (N3)\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}.$$

Besides the above commutative cases, there is a (unique) non-commutative complex Novikov algebra in dimension 2 satisfying equation (2.1):

$$\begin{matrix} (N6) \\ l = -1 \end{matrix}\begin{pmatrix} 0 & e_1 \\ -e_1 & e_2 \end{pmatrix}.$$

Moreover, this algebra is neither associative nor transitive. Hence it is the Novikov algebra satisfying equation (2.1) which is not associative in the lowest dimension.

Example 4. We can obtain the classification of three-dimensional complex Novikov algebras satisfying equation (2.1) from [22]:
Commutative Novikov algebras:

$$(A1)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (A2)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{pmatrix} \quad (A3)\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix} \quad (A4)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & e_1 & e_2 \end{pmatrix}$$

$$(B1)\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix} \quad (B2)\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix} \quad (C1)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix} \quad (C2)\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$$

$$(C11)\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix} \quad (D1)\begin{pmatrix} e_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix} \quad (D2)\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}.$$

Non-commutative associative algebras in example 2

$$(A5)\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & e_1\\ 0 & -e_1 & 0\end{pmatrix} \qquad (A6)\begin{pmatrix}0 & 0 & 0\\ 0 & e_1 & e_1\\ 0 & -e_1 & le_1\end{pmatrix}.$$

Non-associative transitive Novikov algebras satisfying equation (2.1)

$$(A7)\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & e_1\\ 0 & le_1 & e_2\end{pmatrix} \qquad (A8)\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & e_1 & e_2\end{pmatrix}.$$

$$l \neq 1$$

Non-associative and non-transitive Novikov algebras satisfying equation (2.1)

$$\begin{matrix}(B5)\\ l=-1\end{matrix}\begin{pmatrix}0 & 0 & e_1\\ 0 & e_2 & 0\\ -e_1 & 0 & e_3\end{pmatrix} \qquad \begin{matrix}(C5)\\ l=-1\end{matrix}\begin{pmatrix}0 & 0 & e_1\\ 0 & 0 & 0\\ -e_1 & 0 & e_3\end{pmatrix}$$

$$\begin{matrix}(C12)\\ l=-1\end{matrix}\begin{pmatrix}0 & 0 & e_1\\ 0 & 0 & e_2\\ e_1 & -e_2 & e_3\end{pmatrix} \qquad \begin{matrix}(C13)\\ l=-1\end{matrix}\begin{pmatrix}0 & 0 & e_1\\ 0 & 0 & e_2\\ -e_1 & -e_2 & e_3\end{pmatrix}.$$

Moreover, we can see that every three-dimensional transitive Novikov algebra on nilpotent Lie algebras satisfies equation (2.1) [22].

Example 5. There are three four-dimensional nilpotent Lie algebras up to isomorphism: let $\{e_1, e_2, e_3, e_4\}$ be a basis. Then,

$$A = \langle e_1, e_2, e_3, e_4 | [e_i, e_j] = 0\rangle \quad \text{Abelian}$$
$$H = \langle e_1, e_2, e_3, e_4 | [e_2, e_3] = e_1, \text{other products are zero}\rangle$$
$$T = \langle e_1, e_2, e_3, e_4 | [e_2, e_3] = e_1, [e_3, e_4] = e_2, \text{other products are zero}\rangle.$$

The classification of four-dimensional transitive Novikov algebras satisfying equation (2.1) over the real field can be obtained from the classification of four-dimensional real transitive left-symmetric algebras on nilpotent Lie algebras given in [18] (we use the symbols in [18]).

Commutative Novikov algebras:

$$(3)\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & e_1 & 0 & 0\\ 0 & 0 & e_1 & 0\\ 0 & 0 & 0 & e_1\end{pmatrix} \quad (4)\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & e_1 & 0 & 0\\ 0 & 0 & e_1 & 0\\ 0 & 0 & 0 & -e_1\end{pmatrix} \quad (30)_1\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & e_1\\ 0 & 0 & e_1 & 0\\ 0 & e_1 & 0 & e_2\end{pmatrix}$$

$$(41)_{1,1}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & e_1\\ 0 & 0 & e_1 & e_2\\ 0 & e_1 & e_2 & e_3\end{pmatrix} \quad (51)_0\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & e_1 & 0\\ 0 & 0 & 0 & e_1\end{pmatrix} \quad (52)_0\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & e_1 & 0\\ 0 & 0 & 0 & -e_1\end{pmatrix}$$

$$(53)_0\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & e_1 & e_2\\ 0 & 0 & e_2 & -e_1\end{pmatrix} \quad (54)\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & e_1 & 0\\ 0 & 0 & 0 & e_2\end{pmatrix} \quad (57)_0\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & e_1 & e_2\\ 0 & 0 & e_2 & 0\end{pmatrix}$$

$$(60)_1\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & e_1\\ 0 & 0 & e_1 & e_3\end{pmatrix} \quad (61)\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & e_1\end{pmatrix} \quad (62)\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Non-commutative associative algebras in example 2:

$$(5)\begin{pmatrix}0&0&0&0\\0&e_1&e_1&0\\0&-e_1&0&e_1\\0&0&e_1&0\end{pmatrix}\quad(6)\begin{pmatrix}0&0&0&0\\0&0&e_1&0\\0&-e_1&0&e_1\\0&0&e_1&0\end{pmatrix}\quad(7)_t\begin{pmatrix}0&0&0&0\\0&e_1&e_1&0\\0&-e_1&te_1&0\\0&0&0&e_1\end{pmatrix}$$

$$\begin{matrix}(8)_t\\t\geqslant 0\end{matrix}\begin{pmatrix}0&0&0&0\\0&e_1&e_1&0\\0&0&te_1&0\\0&0&0&-e_1\end{pmatrix}\quad(46)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&e_2\\0&0&-e_2&0\end{pmatrix}\quad(47)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_1\\0&0&-e_1&0\end{pmatrix}$$

$$t\geqslant 0$$

$$(48)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_2\\0&0&-e_2&0\end{pmatrix}\quad(49)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_2\\0&0&-e_2&e_1\end{pmatrix}\quad(50)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_2\\0&0&-e_2&-e_1\end{pmatrix}$$

$$\begin{matrix}(51)_t\\t>0\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&te_1\\0&0&-te_1&e_1\end{pmatrix}\quad\begin{matrix}(52)_t\\t>0\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&te_1\\0&0&-te_1&-e_1\end{pmatrix}$$

$$t>0\qquad\qquad t>0$$

$$\begin{matrix}(53)_t\\t>0\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&(1+t)e_2\\0&0&(1-t)e_2&-e_1\end{pmatrix}\quad(55)_t\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_1+te_2\\0&0&-e_1-te_2&e_2\end{pmatrix}$$

$$t>0$$

$$(56)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_1+e_2\\0&0&-e_1+e_2&0\end{pmatrix}\quad\begin{matrix}(57)_t\\t>0\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&(1+t)e_2\\0&0&(1-t)e_2&0\end{pmatrix}.$$

$$t>0$$

Other associative Novikov algebras satisfying equation (2.1):

$$(31)_1\begin{pmatrix}0&0&0&0\\0&0&0&e_1\\0&0&0&e_1\\0&e_1&0&e_2\end{pmatrix}\quad(58)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&e_1&e_3\end{pmatrix}\quad(59)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&e_1\\0&0&e_2&e_3\end{pmatrix}$$

$$\begin{matrix}(60)_t\\t\neq 1\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&e_1\\0&0&te_1&e_3\end{pmatrix}.$$

$$t\neq 1$$

Non-associative Novikov algebras satisfying equation (2.1):

$$(28)\begin{pmatrix}0&0&0&0\\0&0&0&e_1\\0&0&e_1&0\\0&0&0&e_2\end{pmatrix}\quad(29)\begin{pmatrix}0&0&0&0\\0&0&0&e_1\\0&0&0&0\\0&0&e_1&e_2\end{pmatrix}\quad\begin{matrix}(30)_t\\t\neq 1\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&te_1\\0&0&e_1&0\\0&e_1&0&e_2\end{pmatrix}$$

$t \neq 1$

$$\begin{matrix}(31)_t\\t\neq 1\end{matrix}\begin{pmatrix}0&0&0&0\\0&0&0&te_1\\0&0&0&e_1\\0&e_1&0&e_2\end{pmatrix}\quad(40)_1\begin{pmatrix}0&0&0&0\\0&0&0&e_1\\0&0&e_1&e_2\\0&e_1&e_1+e_2&e_3\end{pmatrix}\quad(35)\begin{pmatrix}0&0&0&0\\0&0&0&-e_1\\0&0&0&e_2\\0&e_1&0&e_3\end{pmatrix}.$$

$t \neq 1$

Therefore, we can see that all transitive Novikov algebras on the Lie algebra (H) (the direct sum of a three-dimensional Heisenberg algebra and the field) satisfy equation (2.1) [25]. Moreover, the algebra of type (35) is the only transitive Novikov algebra satisfying equation (2.1) on the Lie algebra (T).

3. Novikov algebras in the case $\tau = 3$

In this section, let A be a Novikov algebra satisfying the following condition:

$$x[y,z] = -[y,z]x \qquad \forall x, y, z \in A. \tag{3.1}$$

Similar to the discussion in section 2, we can let $x = y$ in the above equation:

$$x[x,z] + [x,z]x = 0 \qquad \forall x, z \in A. \tag{3.2}$$

However, the 'linearization' of equation (3.2) by replacing x by $x + y$ cannot lead to equation (3.1), but equation (2.1). In fact, we have the following claim.

Claim 2. *Let A be a Novikov algebra. Then, A satisfies equation (2.2) if and only if A satisfies equation (3.2). That is, equation (3.2) is equivalent to equation (2.2).*

In fact, by the left-symmetry of A, for any $x, z \in A$, we have

$$\begin{aligned}x[x,z] + [x,z]x = 0 &\Leftrightarrow x(xz) - x(zx) + (xz)x - (zx)x = 0\\&\Leftrightarrow x(xz) - z(xx) + [(xz)x - x(zx) + z(xx) - (zx)x] = 0\\&\Leftrightarrow x(xz) = z(xx).\end{aligned}$$

Corollary 2. *Let A be a Novikov algebra. Then A satisfies equation (2.1) if and only if A satisfies*

$$x[y,z] + [y,z]x + y[x,z] + [x,z]y = 0 \qquad \forall x, y, z \in A. \tag{3.3}$$

In fact, the above conclusion can be obtained from the 'linearization' of equation (3.2) by replacing x by $x + y$, or directly, from the left-symmetry of A.

Furthermore, we have the following claim.

Claim 3. *Let A be a Novikov algebra. Then A satisfies equation (3.1) if and only if A satisfies*

$$x(yz) = z(yx) \qquad \forall x, y, z \in A. \tag{3.4}$$

That is, $L_x L_y = R_{yx}$, $\forall x, y \in A$.

In fact, when A satisfies equation (3.1), it satisfies equation (3.3) too. Then we have $\forall x, y, z \in A$,

$$(xy - yx)z = z(xy + yx) - 2y(xz).$$

Since $[x, y]z = -z[x, y]$, *we get* $z(xy) = y(xz)$.

In contrast, if A satisfies equation (3.4), then we have

$$L_{[x,y]} + R_{[x,y]} = L_x L_y - L_y L_x + R_{xy} - R_{yx} = 0 \qquad \forall x, y \in A \tag{3.5}$$

which is equivalent to equation (3.1).

Corollary 3. *Let A be a Novikov algebra satisfying equation (3.1), then A also satisfies equation (2.1). Thus the Novikov algebras in the case* $\tau = 3$ *must be in the case* $\tau = 0$.

Corollary 4. *Let A be a transitive Novikov algebra satisfying equation (3.1), then its sub-adjacent Lie algebra is nilpotent.*

Moreover, from equation (3.5), it is easy to decide whether a Novikov algebra satisfies equation (3.1) from its characteristic matrix: the elements related to the derived ideal $[A, A]$ are anti-symmetric. That is, if $e_{i_1}, \ldots, e_{i_s} \in [A, A]$, then we have

$$c^k_{i_t j} = -c^k_{j i_t} \qquad \forall j, k = 1, \ldots, N \quad \text{and} \quad t = 1, \ldots, s. \tag{3.6}$$

Example 6. From equation (3.6), it is easy to find that all of Novikov algebras appearing in examples 1–5 also satisfy equation (3.1).

Obviously, it seems that the condition equation (3.4) (equation (3.1)) is much stronger than the condition equation (2.2) (equation (2.1)). However, we have not found an example which satisfies equation (2.1) and does not satisfy equation (3.1).

4. Conclusions and discussion

According to the discussion in previous sections, we have the following conclusions:

(1) The Novikov algebras in the case $\tau = 3$ are in the case $\tau = 0$. However, it is an open question whether these two cases coincide, although they coincide in dimension $\leqslant 3$ and for the transitive cases in dimension 4.

(2) We have given the classification of transitive Novikov algebras satisfying equation (2.1) or (3.1) in dimension $\leqslant 4$. That is, we give the classification of (transitive) radicals up to dimension 4. These results will be quite useful for further development including certain applications in physics.

(3) All of the Novikov algebras on three-dimensional Heisenberg Lie algebra satisfying equation (2.1) or (3.1) are transitive. We would like to point out that, in contrast, all of the transitive Novikov algebras on three-dimensional Heisenberg Lie algebra and four-dimensional Heisenberg-type Lie algebra (the Lie algebra (H)) also satisfy equation (2.1) or (3.1). We conjecture that this conclusion may be still true in higher dimensions.

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